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The moduli space of torsion-free G_2 structures

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Abstract

Let $\mathfrak{M}$ be the moduli space of torsion-free G_2 structures on a compact oriented G_2 manifold M. The natural cohomology map $\pi^3 : \mathfrak{M} \to H^3(M, \mathbb{R})$ is known to be a local diffeomorphism [Compact Manifolds with Special Holonomy, Oxford University Press, 2000]. Let $\mathfrak{M}_1 \subset \mathfrak{M}$ be the subset of G_2 structures with volume $(M) = 1$. We show every nonzero element of $H^4(M, \mathbb{R}) = H^3(M, \mathbb{R})^*$ is a Morse function on $\mathfrak{M}_1$ when composed with π^3, and we compute its Hessian. The result implies a special case of Torelli's theorem: if $H^1(M, \mathbb{R}) = 0$ and $\dim H^3(M, \mathbb{R}) = 2$, the cohomology map $\pi^3 : \mathfrak{M} \to H^3(M, \mathbb{R})$ is one to one on each connected component of $\mathfrak{M}$. We formulate a compactness conjecture on the set of G_2 structures of volume $(M) = 1$ with bounded L^2 norm of curvature. If this conjecture were true, it would imply that every connected component of $\mathfrak{M}$ is contractible, and that every compact G_2 manifold supports a G_2 structure whose fundamental 4-form represents the negative of the (nonzero) first Pontryagin class of M. We also observe that when $H^1(M, \mathbb{R}) = 0$, and the volume of the torus $H^3(M, \mathbb{R})/H^3(M, \mathbb{Z})$ is constant along $\mathfrak{M}_1$, the locus $\pi^3(\mathfrak{M}_1) \subset H^3(M, \mathbb{R})$ is a hyperbolic affine sphere.

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1. Introduction

Let $\mathfrak{M}$ be the moduli space of torsion-free G_2 structures on a compact G_2 manifold M. It is known that $\mathfrak{M}$ is a smooth manifold of dimension $b^3 = \dim H^3(M, \mathbb{R})$. When M has full holonomy G_2, or equivalently when $\pi_1(M)$ is finite, a connected component of $\mathfrak{M}$ coincides with the Ricci flat Einstein deformation space of the underlying G_2 metric, as the property for a Ricci flat metric to support a parallel spinor is preserved under Einstein deformation. In particular, $\mathfrak{M}$ is a real analytic manifold endowed with an L^2 Riemannian metric [7,9,10].

The moduli space of polarized Calabi-Yau structures has been studied extensively [13], and there is a recent work of Wilson on the normalized Kähler moduli of Calabi-Yau manifolds [15]. One of our motivation for this work is to propose an extension of these studies to G_2 moduli space, for all of these moduli spaces belong to the category of moduli space of metrics with special holonomy.

Another motivation comes from the following question: *Can one find the best G_2 structure on a given G_2 manifold M?* A natural condition would be to require the fundamental 3-form $\phi \in \Omega^3(M)$ to satisfy $[*_\phi \phi] = -p_1(M)$, where $p_1(M) \in H^4(M, \mathbb{R})$ is the (nonzero) first Pontryagin class and $*_\phi \phi$ is the Hodge dual of ϕ. We will see that such ϕ locally minimizes the L^2 norm of the curvature tensor of the underlying G_2 metric within the set of G_2 structures with fixed volume(M), Section 4.

One of the main results of this paper is the statement that any nonzero element $\beta \in H^4(M, \mathbb{R}) = H^3(M, \mathbb{R})^*$ composed with the cohomology map $\mathfrak{M}_1 \to H^3(M, \mathbb{R})$ is a Morse function on $\mathfrak{M}_1$, where $\mathfrak{M}_1 \subset \mathfrak{M}$ is the subset of the moduli space of G_2 structures of volume 1. Moreover we show that when $b^1 = 0$ and $\beta = -p_1(M) \neq 0$, every critical point is a positive local minimum. As a corollary, we prove a special case of the Torelli theorem; if $b^1 = 0$ and $b^3 = 2$, the cohomology map $\mathfrak{M} \to H^3(M, \mathbb{R})$ is one to one on each connected component of $\mathfrak{M}$.

In order to apply this result to the question of finding a *canonical* G_2 structure, one will need an analogue of Mumford compactness theorem for Riemann surfaces. Based on the geometry of the Torelli map (to be defined below) and the main results above, we formulate in Section 4 a conjecture on the compactness of the set of G_2 structures of volume 1 with uniformly bounded height with respect to $-p_1(M)$. If this compactness theorem is true, a simple Morse theory argument then implies that each connected component of $\mathfrak{M}$ is contractible, and that every compact G_2 manifold M supports a G_2 form ϕ such that $[*_\phi \phi] = -p_1(M)$.

In the last section, we consider the Torelli map $\pi : \mathfrak{M} \to H^3(M, \mathbb{R}) \oplus H^4(M, \mathbb{R})$ defined by

$$\pi(\langle\phi\rangle) = (\pi^3(\langle\phi\rangle), \pi^4(\langle\phi\rangle)) = ([\phi], [*_\phi \phi]) \in H^3(M, \mathbb{R}) \oplus H^4(M, \mathbb{R}),$$

where $\langle\phi\rangle$ represents the equivalence class of G_2 structures represented by ϕ. We prove that if $b^1 = 0$ and $H^3(M, \mathbb{R}) \oplus H^4(M, \mathbb{R})$ is equipped with the canonical metric of signature (b^3, b^4), π is an isometric immersion (up to sign) along $\mathfrak{M}_1$. Moreover, if $b^3 = 2$, the natural Riemannian metric on $\mathfrak{M}$ is flat.

We also describe a G_2 moduli space analogue of Hitchin's result on the canonical embedding of the special Lagrangian moduli space [8]. If $b^1 = 0$ and the volume of the torus $H^3(M,\mathbb{R})/H^3(M,\mathbb{Z})$ is constant along $\mathfrak{M}_1$, the hypersurface $\pi^3(\mathfrak{M}_1) \subset H^3(M,\mathbb{R})$ is a hyperbolic affine sphere centered at the origin. This is also equivalent to the ratio of Jacobians of the projections π^3 and π^4 being constant along $\mathfrak{M}_1$.

2. Moduli space of G_2 structures

In this section we record some definitions and basic results on G_2 structures [9,2,3].

Consider the exterior 3-form φ defined on $\mathbb{R}^7$ by

$$\varphi = (\mathrm{d}x^{12} + \mathrm{d}x^{34} + \mathrm{d}x^{56}) \wedge \mathrm{d}x^7 + \mathrm{d}x^{135} - \mathrm{d}x^{146} - \mathrm{d}x^{362} - \mathrm{d}x^{524}, \tag{1}$$

where x^i's are standard coordinates and $\mathrm{d}x^{12} = \mathrm{d}x^1 \wedge \mathrm{d}x^2$, etc. The stabilizer of φ

$$G_2 = \{A \in GL_7(\mathbb{R}) | A^*\varphi = \varphi\}$$

is the compact, connected, simply connected, rank 2 simple Lie group of dimension 14. G_2 also leaves invariant the 7-form $\mathrm{d}x^{12\cdots7}$ and a metric defined by

$$\langle u, v\rangle\, \mathrm{d}x^{12\cdots7} = \tfrac{1}{6}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi \tag{2}$$

for $u, v \in \mathbb{R}^7$. $GL_7(\mathbb{R})$ orbit of φ is one of two open $GL_7(\mathbb{R})$ orbits in $\bigwedge^3(\mathbb{R}^7)^*$ [2].

Let $\pi : F \to M$ be the principal $GL_7^+(\mathbb{R})$ bundle over a compact oriented 7-manifold M. The fiber over a point $p \in M$ consists of the oriented linear isomorphisms from the tangent space T_pM to $\mathbb{R}^7$. Let $PM = F/G_2 \subset \bigwedge^3 T^*M$ be the open subset whose fiber at each point $p \in M$ consists of $\phi_p \in \bigwedge^3 T_p^*M$ that can be identified with (1) under an oriented isomorphism between T_pM and $\mathbb{R}^7$. An element $\phi \in C^\infty(PM)$ is called a *positive* 3-form. By definition, a positive 3-form ϕ on M is equivalent to a topological G_2 structure on M, i.e., a reduction of the $GL_7^+(\mathbb{R})$ bundle F to a $G_2 \subset SO_7$ bundle

$$\begin{aligned} F_\phi = \{&\mathrm{u} \in F | \mathrm{u} : T_{\pi(\mathrm{u})}M \to \mathbb{R}^7 \text{ is an oriented linear isomorphism such that} \\ &\mathrm{u}^*\varphi = \phi_{\pi(\mathrm{u})}\}. \end{aligned} \tag{3}$$

A compact oriented 7-manifold admits a positive 3-form if and only if it is spin [9].

Let g_ϕ and dvol_ϕ be the metric and the volume form determined by a positive 3-form ϕ as in (2). Since $PM \subset \bigwedge^3 T^*M$ is an open subset, the tangent space $T_\phi C^\infty(PM)$ of the infinite-dimensional manifold $C^\infty(PM)$ is the space of differential 3-forms

$$T_\phi C^\infty(PM) = C^\infty\left(\bigwedge{}^3 T^*M\right) = \Omega^3,$$

and $C^\infty(PM)$ becomes a Riemannian manifold with respect to the L^2 metric on Ω^3 determined by g_ϕ. Note that the diffeomorphism group $\mathrm{Diff}(M)$ acts on $C^\infty(PM)$ by isometries.

Let $*_\phi$ be the Hodge star operator on differential forms induced by ϕ.

Definition 1. A positive 3-form ϕ on an oriented 7-manifold is a G_2 form if

$$\mathrm{d}\phi = 0, \qquad \mathrm{d} *_{\phi} \phi = 0. \tag{4}$$

An oriented 7-manifold is a G_2 manifold if it supports a G_2 form.

Eq. (4) is equivalent to the torsion freeness of the associated $G_2 \subset SO_7$ structure F_ϕ [2]. Let ϕ be a G_2 form on a compact G_2 manifold M. The holonomy of the associated G_2 *metric* g_ϕ is isomorphic to a subgroup of G_2, and g_ϕ is necessarily Ricci-flat. By applying Cheeger–Gromoll splitting theorem for complete Riemannian manifolds with nonnegative Ricci curvature, the holonomy of a G_2 metric is full G_2 whenever the fundamental group $\pi_1(M)$ is finite.

Let $\widehat{\mathfrak{M}} \subset C^\infty(PM)$ denote the space of G_2 forms, and let $\mathcal{D}_0 \subset \mathrm{Diff}(M)$ be the subgroup of diffeomorphisms of M isotopic to the identity.

Definition 2. Let M be a G_2 manifold. The moduli space of G_2 structures (forms) is the quotient space $\mathfrak{M} = \widehat{\mathfrak{M}}/\mathcal{D}_0$.

Given a G_2 form ϕ, we denote its equivalence class by $\langle\phi\rangle$.

Remark 1. The *true* G_2 moduli space is $\widehat{\mathfrak{M}}/\mathrm{Diff}(M)$, and $\mathfrak{M}$ should instead be called the *Teichmúller space* of G_2 structures. However, since only the space $\mathfrak{M}$ will be considered in this paper, we use the term G_2 moduli space for $\mathfrak{M}$.

$\mathfrak{M}$ is a smooth manifold of dimension $b^3 = \dim H^3(M, \mathbb{R})$ [9]. From Remark 1, there exists a Riemannian metric on $\mathfrak{M}$ for which $\widehat{\mathfrak{M}} \to \mathfrak{M}$ is a Riemannian submersion. A connected component of $\mathfrak{M}$ coincides with the Einstein deformation space of the underlying G_2 metrics, which has a real analytic structure [10]. From the definition, the cohomology map $\pi^3 : \mathfrak{M} \to H^3(M, \mathbb{R})$ is well defined, and we denote the image of an equivalence class by $\pi^3(\langle\phi\rangle) = [\phi]$ for simplicity.

Remark 2. Let f be a diffeomorphism of M. Then

$$*_{f^*\phi} f^*\phi = f^*(*_\phi\phi), \tag{5}$$

and the cohomology map $\pi^4 : \mathfrak{M} \to H^4(M, \mathbb{R})$ by $\pi^4(\langle\phi\rangle) = [*_\phi\phi]$ is also well defined. In fact, we may take the dual definition of the moduli space of G_2 structures as the set of equivalence classes of *positive* 4-forms $\psi = *_\phi\phi$ that satisfy $\mathrm{d}\psi = 0$ and $\mathrm{d} *_\psi \psi = 0$. Note however that $*_\phi\phi = *_{(-\phi)}(-\phi)$.

Let $\phi \in \widehat{\mathfrak{M}}$. Then $\bigwedge^* T^*M$ admits a G_2 invariant decomposition

$$\bigwedge{}^2 T^*M = \bigwedge{}^2_7 \oplus \bigwedge{}^2_{14}, \qquad \bigwedge{}^3 T^*M = \bigwedge{}^3_1 \oplus \bigwedge{}^3_7 \oplus \bigwedge{}^3_{27}$$

such that

$$\bigwedge{}^2_7 = \{v \lrcorner \phi | v \in TM\}, \qquad \bigwedge{}^2_{14} = \left\{\eta \in \bigwedge{}^2 T^*M | \eta \wedge *_\phi\phi = 0\right\},$$

$$\bigwedge{}^3_1 = \{\lambda\phi | \lambda \in \mathbb{R}\}, \qquad \bigwedge{}^3_7 = \{v \lrcorner *_\phi \phi | v \in TM\},$$

$$\bigwedge{}^3_{27} = \{h \cdot \phi | h \in S^2(T^*M) \text{ is a quadratic form with } \mathrm{tr}_{g_\phi} h = 0\}, \tag{6}$$

where $h \cdot \phi$ denotes the action of $h \in S^2(T^*M) \subset \mathrm{End}(TM)$ as a derivation. In particular, for any $X \in \Omega^3$ there exists a unique quadratic form h_X and a vector field v_X such that

$$X = h_X \cdot \phi + v_X \lrcorner *_\phi \phi.$$

We set $\Omega^p_k = C^\infty(\bigwedge^p_k)$, and write $X = \sum X_k$ with $X_k \in \Omega^p_k$ for the decomposition of a given $X \in \Omega^p$. Since the Hodge Laplacian commutes with this decomposition [4], it follows that the de Rham cohomology group admits a corresponding Hodge decomposition $H^p(M, \mathbb{R}) = \oplus H^p_k$.

Furthermore $\Omega^3 \cong T_\phi C^\infty(PM)$ admits L^2 orthogonal decomposition along the submanifold $\widehat{\mathfrak{M}} \subset C^\infty(PM)$ as follows, which easily follows from (6) and [9, p. 252].

$$\begin{aligned}
&T_\phi\widehat{\mathfrak{M}} = \{X \in \Omega^3 | \,\mathrm{d}X = 0,\ \mathrm{d} *_\phi (\tfrac{4}{3}X_1 + X_7 - X_{27}) = 0\} = H_\phi \oplus V_\phi,\\
&H_\phi = \{X \in \Omega^3 | \,\mathrm{d}X = 0,\ \mathrm{d} *_\phi X = 0\} = T_\phi\widehat{\mathfrak{M}} \cap V_\phi^\perp,\\
&V_\phi = \{\mathrm{d}\Omega^2_7\}, \qquad N_\phi = (T_\phi\widehat{\mathfrak{M}})^\perp \subset \Omega^3.
\end{aligned} \tag{7}$$

Here $*_\phi((4/3)X_1 + X_7 - X_{27})$ is the derivative of the map $\phi \to *_\phi\phi$. H_ϕ and V_ϕ represent the horizontal and vertical subspaces of $T_\phi\widehat{\mathfrak{M}}$ with respect to the submersion $\widehat{\mathfrak{M}} \to \mathfrak{M}$. The orthogonal projection maps from $T_\phi C^\infty(PM)$ to these subspaces will be denoted by Π^{H}_ϕ, Π^{V}_ϕ, and Π^{N}_ϕ, respectively.

3. Horizontal geodesics on $\widehat{\mathfrak{M}} \to \mathfrak{M}$

Let $\{w^1, w^2, \ldots, w^7\}$ be a local coframe on a G_2 manifold M with a G_2 form ϕ. For a differential 3-form X on M we write,

$$X = \tfrac{1}{6}X_{ijk}w^i \wedge w^j \wedge w^k \in T_\phi C^\infty(PM) \cong \Omega^3,$$

where X_{ijk} is skew symmetric in all of its indices. Then the L^2 inner product on $T_\phi C^\infty(PM)$ is given by

$$\langle\langle X, Y\rangle\rangle_\phi = \int_M \langle X, Y\rangle_\phi \,\mathrm{dvol}_\phi = \frac{1}{6}\int_M X_{ijk}Y_{i'j'k'}g^{ii'}_\phi g^{jj'}_\phi g^{kk'}_\phi \,\mathrm{dvol}_\phi, \tag{8}$$

where $g^{ii'}_\phi = \langle w^i, w^{i'}\rangle_\phi$ represent the inner product on T^*M defined by g_ϕ. Note that $\langle\langle \phi, \phi\rangle\rangle_\phi = 7\,\mathrm{Vol}_\phi(M)$.

Let ∇ be the Levi-Civita connection of the L^2 metric (8) on $C^\infty(PM)$. If we identify tangent vectors to $C^\infty(PM)$ with the Ω^3 valued functions on $C^\infty(PM)$, we have

$$\nabla_X Y = X(Y) + D_X Y, \tag{9}$$

where $X(Y)$ is the directional derivative of Y as an Ω^3 valued function, and $D_X Y$ is the covariant derivative of Y considered as a translation invariant vector field with respect to the natural linear structure of $C^\infty(PM) \subset \Omega^3$. $D_X Y$ can be computed explicitly.

Lemma 1 (Bryant [3]). *Let* $Z = h_Z \cdot \phi + v_Z \lrcorner *_\phi \phi \in T_\phi C^\infty(PM)$, *and consider a curve* $\phi_t = \phi + tZ + \mathrm{O}(t^2) \in \widehat{\mathfrak{M}}$. *Then* $g_{\phi_t} = g_\phi + t2h_Z + \mathrm{O}(t^2)$.

Proposition 1. *Let* $X, Y, Z \in \Omega^3$ *be viewed as translation invariant vector fields in a neighborhood of* $\phi \in C^\infty(PM)$. *Then*

$$Z\langle\langle X, Y\rangle\rangle_\phi = -\langle\langle h_Z \cdot X, Y\rangle\rangle_\phi - \langle\langle X, h_Z \cdot Y\rangle\rangle_\phi + \int_M \mathrm{tr}_{g_\phi}(h_Z)\langle X, Y\rangle_\phi \,\mathrm{dvol}_\phi, \quad (10)$$

and

$$\begin{aligned} 2\langle\langle D_X Y, Z\rangle\rangle_\phi = {} & -2\langle\langle h_X \cdot Y + h_Y \cdot X, Z\rangle\rangle_\phi + \langle\langle h_Z \cdot X, Y\rangle\rangle_\phi + \langle\langle X, h_Z \cdot Y\rangle\rangle_\phi \\ & + \int_M \mathrm{tr}_{g_\phi}(h_X)\langle Y, Z\rangle_\phi \,\mathrm{dvol}_\phi + \int_M \mathrm{tr}_{g_\phi}(h_Y)\langle X, Z\rangle_\phi \,\mathrm{dvol}_\phi \\ & - \int_M \mathrm{tr}_{g_\phi}(h_Z)\langle X, Y\rangle_\phi \,\mathrm{dvol}_\phi. \end{aligned} \quad (11)$$

Proof. For (10), differentiate (8) using Lemma 1. (11) is the standard formula for computing Levi-Civita connection from a Riemannian metric [6], using the fact $[X, Y] = [Y, Z] = [Z, X] = 0$ for they are translation invariant. ☐

Let $\gamma_t \subset \mathfrak{M}$ be a geodesic, and let $\phi_t \subset \widehat{\mathfrak{M}}$ be one of its horizontal lifts based at ϕ_0, which is also a geodesic in $\widehat{\mathfrak{M}}$. As a curve in $C^\infty(PM)$,

$$\Pi_t^{\mathrm{N}}(\phi_t') = 0 \quad (\phi_t \text{ is horizontal}), \quad (12)$$

$$\Pi_t^{\mathrm{N}}(\nabla_{\phi_t'}\phi_t') = \nabla_{\phi_t'}\phi_t' \quad (\phi_t \text{ is geodesic}), \quad (13)$$

where $\Pi_t^{\mathrm{N}} = \Pi_{\phi_t}^{\mathrm{N}}$. Since

$$\nabla_{\phi_t'}\phi_t' = \phi_t'' + D_{\phi_t'}\phi_t',$$

we get from (13)

$$\phi_0'' = \Pi_0^{\mathrm{N}}(\phi_0'') - \Pi_0^{\mathrm{V+H}}(D_{\phi_0'}\phi_0').$$

Differentiating (12)

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\Pi_t^{\mathrm{N}}(\phi_0')\right|_{t=0} + \Pi_0^{\mathrm{N}}(\phi_0'') = 0,$$

and we obtain

$$\phi_0'' = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\Pi_t^{\mathrm{N}}(\phi_0')\right|_{t=0} - \Pi_0^{\mathrm{V+H}}(D_{\phi_0'}\phi_0'). \quad (14)$$

We record the following for later application.

Lemma 2. *Let* $\phi_t \in \widehat{\mathfrak{M}}$ *be a curve, and let* $\psi_0 = *_{\phi_0}\phi_0$. *Let* $X \in \Omega^3$ *be a closed* 3-*form considered as a translation invariant vector field along* ϕ_t. *Then*

$$\int_M \psi_0 \wedge \frac{\mathrm{d}}{\mathrm{d}t}\Pi_t^{\mathrm{N}}(X)\bigg|_{t=0} = 0.$$

Proof. Write $X = X_t^{\mathrm{H}} + X_t^{\mathrm{V}} + X_t^{\mathrm{N}}$, where $X_t^{\mathrm{H}} \in H_{\phi_t}$, $X_t^{\mathrm{V}} \in V_{\phi_t}$, and $X_t^{\mathrm{N}} \in N_{\phi_t}$. From the decomposition (7), $\mathrm{d}X = \mathrm{d}X_t^{\mathrm{N}} = 0$. Since N_{ϕ_t} is orthogonal to H_{ϕ_t}, the space of harmonic forms, every closed element in N_{ϕ_t} is in fact exact. $\square$

4. Morse functions on $\mathfrak{M}_1$

Let $\widehat{\mathfrak{M}}_1 \subset \widehat{\mathfrak{M}}$ be the set of G_2 forms with $\mathrm{Vol}_\phi(M) = 1$. Since $\mathcal{D}_0$ acts trivially on the top cohomology $H^7(M, \mathbb{R})$, there exists an induced action of $\mathcal{D}_0$ on $\widehat{\mathfrak{M}}_1$. We let $\mathfrak{M}_1 = \widehat{\mathfrak{M}}_1/\mathcal{D}_0 \subset \mathfrak{M}$ be the subset of equivalence classes of G_2 structures of volume 1. Then $\mathfrak{M} \cong \mathfrak{M}_1 \times \mathbb{R}^+$ naturally, and $\mathfrak{M}_1$ is an embedded hypersurface of $\mathfrak{M}$. In this section, we show that every nonzero element in $H^4(M, \mathbb{R}) = H^3(M, \mathbb{R})^*$ is a Morse function when composed with the cohomology map $\mathfrak{M}_1 \to H^3(M, \mathbb{R})$, and we compute its Hessian.

Define the volume function

$$V(\phi) = \frac{1}{7}\int_M \phi \wedge *_\phi\phi : \widehat{\mathfrak{M}} \to \mathbb{R}^+. \tag{15}$$

For $X = X_1 + X_7 + X_{27} \in T_\phi\widehat{\mathfrak{M}}$,

$$\begin{aligned}\nabla_X V &= \frac{1}{7}\int_M X_1 \wedge *_\phi\phi + \phi \wedge \frac{4}{3} *_\phi X_1 \quad (\text{by (7)})\\ &= \frac{1}{3}\int_M X_1 \wedge *_\phi\phi = \frac{1}{3}\langle\langle X, \phi\rangle\rangle_\phi.\end{aligned}$$

Thus the gradient of the volume function at ϕ is

$$\nabla V = \tfrac{1}{3}\phi \in H_\phi,$$

and we denote $\nu_\phi = (1/\sqrt{7})\phi \in H_\phi$ the unit normal to $\widehat{\mathfrak{M}}_1 \subset \widehat{\mathfrak{M}}$ at ϕ. The second fundamental form of the hypersurface $\widehat{\mathfrak{M}}_1$ is then

$$\mathrm{II} = -\langle\langle \delta\nu, \delta\phi\rangle\rangle = -\frac{1}{\sqrt{7}}\langle\langle \delta\phi, \delta\phi\rangle\rangle,$$

and $\widehat{\mathfrak{M}}_1 \subset \widehat{\mathfrak{M}}$ is a umbilic hypersurface. Here $\delta\phi$ in the expression above is a tautological 1-form that represents the infinitesimal displacement of ϕ in the vector space $\widehat{\mathfrak{M}}$.

Next we introduce a special class of functions on $\mathfrak{M}$. For $\beta \in H^4(M, \mathbb{R}) = H^3(M, \mathbb{R})^*$, define

$$F^\beta(\langle\phi\rangle) = \beta([\phi]). \tag{16}$$

Let F_1^β denote its restriction to $\mathfrak{M}_1$, and $\mathrm{Crit}(F_1^\beta)$ denote its critical set.

Remark 3. From Remark 2, $G^{\alpha}(\langle\phi\rangle) = \alpha([*_{\phi}\phi])$ is also well defined on $\mathfrak{M}$ for $\alpha \in H^3(M, \mathbb{R}) = H^4(M, \mathbb{R})^*$.

Proposition 2.

$$\mathrm{Crit}(F_1^{\beta}) = \left\{ \langle\phi\rangle | \beta = c^{\beta}_{\langle\phi\rangle}[*_{\phi}\phi] \ \textit{for some constant}\ c^{\beta}_{\langle\phi\rangle} = \frac{1}{7}F_1^{\beta}(\langle\phi\rangle) \right\}.$$

Proof. From the description of the unit normal ν above, $\langle\phi\rangle$ is a critical point of F_1^{β} whenever β annihilates $H_7^3 \oplus H_{27}^3 \subset H_{\phi}$. The proposition follows for $H^3(M, \mathbb{R})^* = H^4(M, \mathbb{R})$. □

Let $\langle\phi_0\rangle \in \mathrm{Crit}(F_1^{\beta})$ and put $\psi_0 = *_{\phi_0}\phi_0$. Then one finds

$$\nabla^2 F_1^{\beta}|_{\langle\phi_0\rangle} = \nabla^2 F^{\beta}|_{\langle\phi_0\rangle} + \frac{\partial}{\partial\nu}F^{\beta}\ |_{\langle\phi_0\rangle}\mathrm{II}_{\langle\phi_0\rangle} = \nabla^2 F^{\beta}|_{\langle\phi_0\rangle} - c^{\beta}_{\langle\phi_0\rangle}\langle\langle\delta\phi, \delta\phi\rangle\rangle_{\langle\phi_0\rangle}. \quad (17)$$

Here we continue to use ∇ to denote the Levi-Civita connection of the Riemannian manifold $\mathfrak{M}$. Let $X \in H_{\phi_0}$ be a horizontal lift of a tangent vector $x \in T_{\langle\phi_0\rangle}\mathfrak{M}_1$, and let $\phi_t \subset \widehat{\mathfrak{M}}$ be a horizontal geodesic with $\phi_0' = X$. From (14) and Lemma 2,

$$\nabla^2 F^{\beta}(x, x) = \int_M c^{\beta}_{\langle\phi_0\rangle}\psi_0 \wedge \phi_0'' = -c^{\beta}_{\langle\phi_0\rangle}\langle\langle\phi_0, D_X X\rangle\rangle_{\phi_0}.$$

Set $X = v_X \lrcorner \psi_0 + h_X \cdot \phi_0 = X_7 + X_{27}$. Proposition 1 then gives,

$$\langle\langle\phi_0, D_X X\rangle\rangle_{\phi_0} = -2\langle\langle h_X \cdot X, \phi_0\rangle\rangle_{\phi_0} + \langle\langle X, X\rangle\rangle_{\phi_0} - \tfrac{7}{6}\langle\langle X, X\rangle\rangle_{\phi_0}$$

(since $h_{\phi_0} = \frac{1}{3}g_{\phi_0}$ and $X \in T_{\phi_0}\widehat{\mathfrak{M}}_1$)

$$= -\tfrac{1}{6}\langle\langle X, X\rangle\rangle_{\phi_0} - 2\langle\langle h_X \cdot X, \phi_0\rangle\rangle_{\phi_0}$$

$$= -\tfrac{1}{6}\langle\langle X, X\rangle\rangle_{\phi_0} - 2\langle\langle X_{27}, X_{27}\rangle\rangle_{\phi_0}.$$

Theorem 1. *Let* F_1^{β} *be the function on* $\mathfrak{M}_1$ *defined by* (16) *for* $\beta \in H^4(M, \mathbb{R})$. *The Hessian of* F_1^{β} *at a critical point* $\langle\phi_0\rangle \in \mathfrak{M}_1$ *is*

$$\nabla^2 F_1^{\beta}|_{\langle\phi_0\rangle}(x, x) = -c^{\beta}_{\langle\phi_0\rangle}(\tfrac{5}{6}\langle\langle x_7, x_7\rangle\rangle_{\langle\phi_0\rangle} - \tfrac{7}{6}\langle\langle x_{27}, x_{27}\rangle\rangle_{\langle\phi_0\rangle}),$$

where $x = x_7 + x_{27} \in T_{\langle\phi_0\rangle}\mathfrak{M}_1 = H_7^3 \oplus H_{27}^3$, $\beta = c^{\beta}_{\langle\phi_0\rangle}[*_{\phi_0}\phi_0]$ *with* $c^{\beta}_{\langle\phi_0\rangle} = (1/7)F_1^{\beta}(\langle\phi_0\rangle)$. *In particular* F_1^{β} *is a Morse function on* $\mathfrak{M}_1$ *for any nonzero* $\beta \in H^4(M, \mathbb{R})$.

Note that if $H^1(M, \mathbb{R}) = 0$, every critical point of F_1^{β} is either a positive local minimum or a negative local maximum.

Let $p_1(M) \in H^4(M, \mathbb{R})$ be the first Pontryagin class of a G_2 manifold M. It is known for any G_2 form $\phi \in \widehat{\mathfrak{M}}_1$

$$-p_1(M)([\phi]) = \tfrac{1}{56}\pi^2\|R_{g_\phi}\|^2, \quad (18)$$

where $\|R_{g_\phi}\|$ is the L^2 norm of the curvature tensor of the G_2 metric g_ϕ [9].

Corollary 1. *Let M be a compact oriented* G_2 *manifold, and let* $\mathfrak{M}_1$ *be the moduli space of* G_2 *structures of volume* 1. *Then each connected component of* $\mathfrak{M}_1$ *is noncompact except when* $\dim H^3(M, \mathbb{R}) = 1$.

Proof. Suppose $p_1(M) = 0$. Then by (18), any G_2 structure on M is necessarily flat. M must then be a quotient of a flat torus, and $\mathfrak{M}_1$ is a $SO_7/G_2 = \mathbb{R}P^7$ bundle over the moduli space of flat Riemannian metrics on M with volume 1, which is noncompact.

Suppose $p_1(M) \neq 0$, then $F_1^{-p_1(M)} > 0$ on $\mathfrak{M}_1$. The corollary follows from Theorem 1 by maximum principle. □

Theorem 1 also implies the following special case of the Torelli theorem for G_2 structures. Set $b^1 = \dim H^1(M, \mathbb{R})$ and $b^3 = \dim H^3(M, \mathbb{R})$.

Corollary 2. *Let M be a compact oriented* G_2 *manifold with* $b^1 = 0$ *and* $b^3 = 2$. *Then each of the cohomology maps* $\pi^3 : \mathfrak{M} \to H^3(M, \mathbb{R})$ *and* $\pi^4 : \mathfrak{M} \to H^4(M, \mathbb{R})$ *is one to one on every connected component of* $\mathfrak{M}$.

Proof. Let $\phi_0, \phi \in \widehat{\mathfrak{M}_1}$ and put $\psi_0 = *_{\phi_0}\phi_0$, $\psi = *_\phi\phi$. Suppose π^4 is not one to one and, since $\mathfrak{M} = \mathfrak{M}_1 \times \mathbb{R}^+$, assume $[\psi] = \lambda[\psi_0]$ for a constant $\lambda \neq 0$. Then both $\langle\phi\rangle$ and $\langle\phi_0\rangle$ are critical points of $F_1^{[\psi_0]}$ by Proposition 2. But since $b^1 = 0$ and $\dim \mathfrak{M}_1 = 1$, $F_1^{[\psi_0]}$ is positive and its critical points can only be local minima. Therefor $F_1^{[\psi_0]}$ must be a positive convex function on $\langle\phi_0\rangle$ component of $\mathfrak{M}_1$ with only one critical point $\langle\phi_0\rangle$.

Suppose π^3 is not one to one and assume $[\phi] = \lambda[\phi_0]$. The result follows from the similar argument by considering $G^{[\phi_0]}$, see Remark 3 for definition. □

Based on Theorem 1 and Corollary 2, we propose the following conjecture on the compactness of G_2 structures of volume 1 with bounded L^2 norm of curvature.

Conjecture 1. Let $\{\phi_n\}$ be a sequence of G_2 forms on a compact oriented 7-manifold M with $H^1(M, \mathbb{R}) = 0$ such that $\mathrm{Vol}_{\phi_n}(M) = 1$. Suppose $\{-p_1(M)([\phi_n])\} = \{(1/56\pi^2)\|R_{g_{\phi_n}}\|^2\}$ is bounded from above. Then there exists a subsequence $\{\phi_{n_k}\}$, a sequence of diffeomorphisms f_k, and a G_2 form ϕ such that $f_k^*\phi_{n_k} \to \phi$ in C^1.

Suppose this conjecture is true, and consider the function $F_1^{-p_1(M)}$. From the C^1 convergence we have $[f_k^*\phi_{n_k}] \to [\phi]$ and $[f_k^* *_{\phi_{n_k}} \phi_{n_k}] \to [*_\phi\phi]$, and the conjecture implies that the set

$$(F_1^{-p_1(M)})^c = \{\langle\phi\rangle \in \mathfrak{M}_1 | F_1^{-p_1(M)}(\langle\phi\rangle) \leq c\}$$

is compact for any constant c. Hence $F_1^{-p_1(M)}$ is a positive proper function [11]. Since every critical points of $F_1^{-p_1(M)}$ is a positive local minimum, a result form Morse theory [11, p. 20] shows that every connected component of $\mathfrak{M}_1$ is contractible, and that there exists a unique critical point $\langle\phi_0\rangle$ of $F_1^{-p_1(M)}$ in each connected component of $\mathfrak{M}_1$. By Theorem 1, $[*_{\phi_0}\phi_0]$ must then be a constant multiple of $-p_1(M)$. Since $\mathfrak{M} \cong \mathfrak{M}_1 \times \mathbb{R}^+$, this implies every connected component of $\mathfrak{M}$ is contractible and contains a unique element $\langle\phi_0\rangle$ such that $[*_{\phi_0}\phi_0] = -p_1(M)$.

5. The Torelli map

Definition 3. Let $\mathfrak{M}$ be the moduli space of G_2 structures on a compact G_2 manifold M. The Torelli map $\pi : \mathfrak{M} \to H^3(M, \mathbb{R}) \oplus H^4(M, \mathbb{R})$ is defined by

$$\pi(\langle\phi\rangle) = (\pi^3(\langle\phi\rangle), \pi^4(\langle\phi\rangle)) = ([\phi], [*_\phi\phi]).$$

The purpose of this section is to describe the image of the Torelli map $\pi(\mathfrak{M})$ and the hypersurface $\pi^3(\mathfrak{M}_1) \subset H^3(M, \mathbb{R})$ (or equivalently $\pi^4(\mathfrak{M}_1) \subset H^4(M, \mathbb{R})$). A general idea is that any invariants of $\pi(\mathfrak{M})$ as a Lagrangian submanifold, see (21), or the invariants of $\pi^3(\mathfrak{M}_1)$ as an affine hypersurface will be invariants of $\mathfrak{M}$. We assume in this section $H^1(M, \mathbb{R}) = 0$.

Let $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ be a basis of $H^3(M, \mathbb{R})$ and $\{\beta^1, \beta^2, \dots, \beta^l\}$ be its dual basis of $H^4(M, \mathbb{R})$ so that $\beta^A(\alpha_B) = \delta^A_B$, where $l = b^3 = \dim H^3(M, \mathbb{R})$. Let $x^t = (x^1, x^2, \dots, x^l)$ and $y^t = (y_1, y_2, \dots, y_l)$ be the coordinates of $H^3(M, \mathbb{R})$ and $H^4(M, \mathbb{R})$ with respect to these basis. Then $\sum_A \mathrm{d}x^A \wedge \mathrm{d}y_A$ is the canonical symplectic form and $G = \sum_A \mathrm{d}x^A\, \mathrm{d}y_A$ is the canonical metric of signature (l, l) on $H^3(M, \mathbb{R}) \oplus H^4(M, \mathbb{R}) = H^3(M, \mathbb{R}) \oplus H^3(M, \mathbb{R})^*$. By definition,

$$\pi^3(\langle\phi\rangle) = \sum_A x^A(\langle\phi\rangle)\alpha_A, \qquad \pi^4(\langle\phi\rangle) = \sum_A y_A(\langle\phi\rangle)\beta^A, \tag{19}$$

where $x^A(\langle\phi\rangle) = \beta^A([\phi])$, $y_A(\langle\phi\rangle) = \alpha_A([*_\phi\phi])$. Each x and y is a local diffeomorphism from $\mathfrak{M}$ to $\mathbb{R}^l$ [9]. The volume function (15) is well defined on $\mathfrak{M} = \widehat{\mathfrak{M}}/\mathfrak{D}_0$, and in terms of these coordinates is given by the formula

$$V(\langle\phi\rangle) = \frac{1}{7}\int_M \phi \wedge *_\phi\phi = \frac{1}{7}\sum_A x^A y_A,$$

and

$$\begin{aligned} \mathrm{d}V &= \frac{1}{7}\sum_A x^A\, \mathrm{d}y_A + y_A\, \mathrm{d}x^A = \frac{1}{7}\int_M \phi \wedge \mathrm{d}\pi^4 + \mathrm{d}\pi^3 \wedge *_\phi\phi. \\ &= \frac{1}{3}\int_M \mathrm{d}\pi^3 \wedge *_\phi\phi \quad (\text{by (7)}) \\ &= \frac{1}{3}\sum_A y_A\, \mathrm{d}x^A. \end{aligned} \tag{20}$$

Hence

$$3\sum_A x^A\, \mathrm{d}y_A = 4\sum_A y_A\, \mathrm{d}x^A, \tag{21}$$

and the Torelli map π is a Lagrangian immersion [9].

Let $p=(p_{AB})=(p_{BA})$ be the unique $GL_l(\mathbb{R})$ valued function on $\mathfrak{M}$ such that $\mathrm{d}y_A=\sum_B p_{AB}\,\mathrm{d}x^B$. From (21), we have

$$4y=3px, \tag{22}$$

and the invertible symmetric matrix function p transforms the period function x to y. Upon a change of coordinates $x^*=a^{-1}x$, $y^*=a^t y$ for $a\in GL_l(\mathbb{R})$, p becomes $p^*=a^t pa$, and $\mathrm{d}x^t\,p\,\mathrm{d}x=\mathrm{d}x^t\cdot\mathrm{d}y$ is a well-defined quadratic form on $\mathfrak{M}$, which is the pulled back metric $\pi^*(G)$.

Proposition 3. *The signature of* π^*G *is* $(1,b^3-1)$. $-\pi^*G$ *is the* L^2 *Riemannian metric when restricted to* $\mathfrak{M}_1\subset\mathfrak{M}$.

Proof. *Let* $Z=Z_1+Z_{27}\in H_\phi$ *be the horizontal lift of* $z=z_1+z_{27}\in T_{\langle\phi\rangle}\mathfrak{M}_1$. *Then*

$$\begin{aligned}\pi^*G(z,z)&=\mathrm{d}x^t(z)\cdot\mathrm{d}y(z)=\tfrac{4}{3}\langle\langle Z_1,Z_1\rangle\rangle_\phi-\langle\langle Z_{27},Z_{27}\rangle\rangle_\phi\quad(\text{by }(7))\\&=\tfrac{4}{3}\langle\langle z_1,z_1\rangle\rangle_{\langle\phi\rangle}-\langle\langle z_{27},z_{27}\rangle\rangle_{\langle\phi\rangle}.\end{aligned}$$

□

Let $\{\xi_1(\phi),\xi_2(\phi),\ldots,\xi_l(\phi)\}$ be a basis of harmonic 3-forms with respect to the G_2 metric g_ϕ such that $[\xi_A(\phi)]=\alpha_A$. The L^2 Riemannian metric tensor on $\mathfrak{M}$ can be written as $m_{AB}(\langle\phi\rangle)\,\mathrm{d}x^A\,\mathrm{d}x^B$ where by definition

$$m_{AB}(\langle\phi\rangle)=\int_M *_\phi\xi_A(\phi)\wedge\xi_B(\phi)=\langle\langle\xi_A(\phi),\xi_B(\phi)\rangle\rangle_\phi \tag{23}$$

or equivalently

$$[*_\phi\xi_A(\phi)]=\sum_B m_{AB}(\langle\phi\rangle)\beta^B.$$

From Proposition 3 and (20), (22), it easily follows that

$$m_{AB}=-p_{AB}+\frac{1}{3V}y_Ay_B,\qquad m^{AB}=-p^{AB}+\frac{1}{4V}x^Ax^B, \tag{24}$$

where (m^{AB}) and (p^{AB}) are inverse matrices of (m_{AB}) and (p_{AB}), respectively. (24) and Eqs. (21), (22) for example can be used to show $\Delta V=(7/18)b^3>0$.

Remark 4. When $b^3=2$, a computation using (24) shows $m_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B$ is a flat Riemannian metric on $\mathfrak{M}$.

We now turn our attention to the hypersurface $\Sigma=[\mathfrak{M}_1]\subset H^3(M,\mathbb{R})$. For concreteness, let us assume that $\{\alpha_1,\alpha_2,\ldots,\alpha_l\}$ is a basis of $H^3(M,\mathbb{Z})/$ torsion. Choose an orientation for $H^3(M,\mathbb{R})$, and we consider the properties of Σ that are invariant under the linear change of basis by $SL_l(\mathbb{R})$ [5,12]. We agree on the index range $1\le i,j\le l-1$ and $1\le A,B\le l$.

Let $\vec{x}=\sum_A x^A\alpha_A$ be the immersion defined by (19). Since $\mathfrak{M}_1$ is defined by the equation $V=1$, we may write

$$\begin{aligned}\mathrm{d}\vec{x}&=\sum_i\mathrm{d}x^i\,\alpha_i+\mathrm{d}x^l\,\alpha_l=\sum_i\mathrm{d}x^i\left(\alpha_i-\frac{y_i}{y_l}\alpha_l\right)+\left(\mathrm{d}x^l+\frac{1}{y_l}\sum_i y_i\,\mathrm{d}x^i\right)\alpha_l\\&=\sum_i\omega^ie_i+\omega^le_l,\end{aligned} \tag{25}$$

where

$$e_i = \alpha_i - \frac{y_i}{y_l}\alpha_l, \qquad e_l = \alpha_l, \tag{26}$$

and $\omega^i = \mathrm{d}x^i$. Note that by (25) we have $\omega^l = \mathrm{d}x^l + (1/y_l)\sum_i y_i\,\mathrm{d}x^i = 0$. Also after a linear change of coordinates if necessary, we may assume that $y_l > 0$ and hence $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \mathrm{d}x^{l-1} \neq 0$. Following the general theory of moving frames, define (ω_A^B) by

$$\mathrm{d}e_A = \sum_B \omega_A^B e_B.$$

Differentiating (26), we get

$$\mathrm{d}e_i \equiv \omega_i^l e_l \mod e_1, e_2, \ldots, e_{l-1} \equiv -\mathrm{d}\left(\frac{y_i}{y_l}\right) e_l, \tag{27}$$

and

$$\begin{aligned}
\omega_i^l &= -\mathrm{d}\left(\frac{y_i}{y_l}\right) = -\frac{p_{ij}}{y_l}\,\mathrm{d}x^j - \frac{p_{il}}{y_l}\,\mathrm{d}x^l + \frac{y_i}{y_l^2}\,\mathrm{d}y_l \\
&= -\frac{p_{ij}}{y_l}\,\mathrm{d}x^j + \frac{1}{y_l^2}\left((y_i p_{lj} + y_j p_{li})\,\mathrm{d}x^j - \frac{p_{ll}}{y_l} y_i y_j\,\mathrm{d}x^j\right) \quad (\text{since}\,\omega^l = 0) \\
&= \left(-\frac{p_{ij}}{y_l} + \frac{p_{li}y_j + p_{lj}y_i}{y_l^2} - \frac{p_{ll}}{y_l^3} y_i y_j\right)\mathrm{d}x^j = h_{ij}\omega^j.
\end{aligned}$$

The second fundamental form of Σ is by definition

$$\mathrm{II} = \sum_{ij} h_{ij}\omega^i\omega^j = -\frac{1}{y_l}\sum_{AB} p_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B.$$

Since we are assuming $H^1(M, \mathbb{R}) = 0$, II is definite by Proposition 3 and we may set $H = \det(h_{ij}) > 0$. The normalized second fundamental form

$$\widehat{\mathrm{II}} = H^{-1/(l+1)}\mathrm{II}$$

is an affine invariant called *Blaschke metric* [5]. $\omega_1^l \wedge \omega_2^l \cdots \wedge \omega_{l-1}^l = H\omega^1 \wedge \omega^2 \cdots \wedge \omega^{l-1}$ by definition, and a short computation using (27) gives

$$\begin{aligned}
\omega_1^l \wedge \omega_2^l \cdots \wedge \omega_{l-1}^l &= \frac{1}{y_l^l}\left(\sum_A y_A \frac{\partial}{\partial y_A} \lrcorner \mathrm{d}y_1 \wedge \mathrm{d}y_2 \cdots \wedge \mathrm{d}y_l\right) \\
&= \frac{1}{y_l^l}\frac{3}{4}\det(p)\left(\sum_A x^A \frac{\partial}{\partial x^A} \lrcorner \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^l\right) \\
&= \frac{1}{y_l^l}\frac{3}{4}\det(p)(-1)^{l-1}\frac{\sum_A x^A y_A}{y_l}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^{l-1}
\end{aligned}$$

for $\sum_A y_A(\partial/\partial y_A) = (3/4)\sum_A x^A(\partial/\partial x^A)$ by (21). Since $V = (1/7)\sum_A x^A y_A = 1$,

$$H = \tfrac{21}{4}\det(p)(-1)^{l-1}y_l^{-(l+1)},$$

and

$$\widehat{\mathrm{II}} = -\left(\frac{21}{4}\det(p)(-1)^{l-1}\right)^{-1/(l+1)}\sum_{AB} p_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B.$$

Another affine invariant $\vec{\xi}^x$, called *affine normal*, is defined by

$$\vec{\xi}^x = H^{1/(l+1)}\left(e_l + \sum_i t^i e_i\right),$$

where t^i's are uniquely determined by the equation

$$\frac{1}{l+1}\,\mathrm{d}\log|H| + \sum_i t^i\omega_i^l = 0 = \lambda\sum_A x^A\,\mathrm{d}y_A$$

for some nonzero λ in our case [5]. Set $q^A = (1/(l+1)\det(p))(\partial\det(p)/\partial y_A)$. Then a simple computation shows that $t^i = y_l(q^i - x^i\lambda)$, where $\lambda = (1/7)(-1 + \sum_A y_A q^A)$. The affine normal $\vec{\xi}^x$ is now determined to be

$$\begin{aligned}\vec{\xi}^x &= |H|^{1/(l+1)}\left(e_l + \sum_i t^i e_i\right)\\ &= |H|^{1/(l+1)}\left(e_l + \frac{y_l}{7}x^i e_i + y_l\left(q^i - \frac{x^i}{7}\left(\sum_B y_B q^B\right)\right)e_i\right)\\ &= |H|^{1/(l+1)}\frac{y_l}{7}\left(x^A\alpha_A + \left(7q^A - x^A\left(\sum_B y_B q^B\right)\right)\alpha_A\right).\end{aligned} \tag{28}$$

A hypersurface in an affine space is called an *affine sphere* if all the affine normal lines pass through a fixed point(finite or infinite) called the center. In case it is locally convex, it is called elliptic or hyperbolic depending on whether the center is on the convex or concave side.

Theorem 2. *Let M be a compact G_2 manifold with $H^1(M,\mathbb{R}) = 0$. The locus of projection $\pi^3(\mathfrak{M}_1) \subset H^3(M,\mathbb{R})$ is a hyperbolic affine sphere centered at the origin iff* $\det(p_{AB})$ (*equivalently* $\det(m_{AB})$) *is constant along* $\mathfrak{M}_1$.

Proof. From the formula (28), $\vec{\xi}^x$ is proportional to $\vec{x} = x^A\alpha_A$ iff $q^A = \mu x^A$ on $\mathfrak{M}_1$ for some function μ for all A. By definition of q^A, this is equivalent to $\mathrm{d}\det(p_{AB}) = 0$ on $\mathfrak{M}_1$.

From (24), we have

$$\begin{aligned}\sum_{AB} m^{AB}\,\mathrm{d}m_{AB} &\equiv \sum_{AB} p^{AB}\,\mathrm{d}p_{AB} - \frac{1}{3V}p^{AB}(y_A\,\mathrm{d}y_B + y_B\,\mathrm{d}y_A)\\ &\quad - \frac{1}{4V}p_{AB}(x^A\,\mathrm{d}x^B + x^B\,\mathrm{d}x^A)\\ &\quad + \frac{1}{12V^2}x^A x^B(y_A\,\mathrm{d}y_B + y_B\,\mathrm{d}y_A) \quad \mathrm{mod} \quad \mathrm{d}V \quad (\text{by}\,(21),\ (22))\\ &\equiv \sum_{AB} p^{AB}\,\mathrm{d}p_{AB} \quad \mathrm{mod} \quad \mathrm{d}V.\end{aligned}$$

$\pi^3(\mathfrak{M}_1)$ is locally convex by Proposition 3 and hyperbolic by Theorem 1. □

By definition, $\det(m_{AB})$ is constant on $\mathfrak{M}_1$ iff the volume of the torus $H^3(M,\mathbb{R})/H^3(M,\mathbb{Z})$ is constant on $\mathfrak{M}_1$. Theorem 2 is analogous to Hitchin's result on the canonical embedding of the special Lagrangian moduli [8].

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